6．3　積分公式の誤差評価

テイラー級数による誤差評価

R は $\int_a^b g(x)dx$ 及び $F[g(x)]$ のテイラー展開の差により求められる

$$R = \int_a^b g(x)dx - F[g(x)] \qquad (b - a = h)$$

積分の真値

積分公式

$$g(x) = G'(x)$$

$$\int_a^b g(x)dx = G(b) - G(a)$$

台形公式の誤差の誤差評価式の導出

台形公式: $$F[g(x)] = \frac{h}{2}[g(a) + g(b)]$$

$$R = \int_a^b g(x)dx - F[g(x)]$$

$$\int_a^b g(x)dx = G(b) - G(a)$$

$$G(b) = G(a+h) = G(a) + \frac{h}{1!}G'(a) + \frac{h^2}{2!}G''(a) + \frac{h^3}{3!}G'''(a) + \cdots$$

$$g(x) = G'(x)$$

$$\int_a^b g(x)dx = hg(a) + \frac{h^2}{2!}g'(a) + \frac{h^3}{3!}g''(a) + \cdots$$

台形公式の誤差の誤差評価式の導出

台形公式：

$$F[g(x)] = \frac{h}{2}[g(a) + g(b)]$$

$$g(b) = g(a+h) = g(a) + \frac{h}{1!}g'(a) + \frac{h^2}{2!}g''(a) + \frac{h^3}{3!}g'''(a) + \cdots$$

$$F[g(x)] = \frac{h}{2}\left[2g(a) + hg'(a) + \frac{h^2}{2!}g''(a) + \cdots\right]$$

$$\int_a^b g(x)dx = hg(a) + \frac{h^2}{2!}g'(a) + \frac{h^3}{3!}g''(a) + \cdots$$

$$R = -\frac{h^3}{12}g''(a) - \frac{h^4}{24}g'''(a) - \frac{h^5}{80}g^{(4)}(a) - \cdots$$

多項式による誤差評価

$F[g(x)]$ が成り立つ最高次の次数を $n=N$ とすると

$$R = \int_a^b g(x)dx - F[g(x)]$$

$$R_N = \int_a^b \frac{x^{N+1}}{(N+1)!} g^{(N+1)}(\xi)dx - F\left[\frac{x^{N+1}}{(N+1)!} g^{(N+1)}(\xi)\right]$$

$$R_N \le \frac{1}{(N+1)!}\left| g_{max}^{(N+1)}\left[\frac{b^{N+2} - a^{N+2}}{N+2} - F\left(x^{N+1}\right)\right]\right|$$

シンプソンの1/3則の誤差の導出

$$R_N \le \frac{1}{(N+1)!}\left| g_{max}^{(N+1)} \left[\frac{b^{N+2} - a^{N+2}}{N+2} - F\left(x^{N+1}\right) \right] \right|$$

3次の多項式　$N = 2$　$x_0 = 0, x_1 = h, x_2 = 2h$

$$\int_0^{2h} x^3 dx \cong F\left(x^3\right) = \frac{h}{3}\left[0^3 + 4h^3 + (2h)^3\right] = 4h^4$$

$$R_2 \le \frac{1}{3!}\left| g_{\max}^{(3)} \left[\frac{(2h)^4 - 0^4}{4} - 4h^4 \right] \right| = 0$$

シンプソンの1/3則の誤差の導出

$$R_N \le \frac{1}{(N+1)!}\left| g_{max}^{(N+1)} \left[\frac{b^{N+2} - a^{N+2}}{N+2} - F\left(x^{N+1}\right) \right] \right|$$

4次の多項式　$N = 3$　$x_0 = 0,\ x_1 = h,\ x_2 = 2h$

$$\int_0^{2h} x^4 dx \cong F\left(x^4\right) = \frac{h}{3}\left[0^4 + 4h^4 + (2h)^4\right] = \frac{20}{3}h^5$$

$$R_3 \le \frac{1}{4!}\left| g_{max}^{(4)} \left[\frac{(2h)^5 - 0^5}{5} - \frac{20}{3}h^5 \right] \right| = \frac{1}{90}h^5 \left| g_{max}^{(4)} \right|$$

$$4 \times 3 \times 2 \times 1$$

$$\frac{32}{5} - \frac{20}{3} = \frac{96 - 100}{15}$$